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CONSTRUCTIVE CONTROL OF THE MOTION OF OSCILLATING SYSTEMS WITH DISCRETE AND DISTRIBUTED PARAMETERS*

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The problem of controlling the motion of mechanical systems over an asymptotically large but fixed time interval is considered. The controlled objects may include elements with discrete parameters (material points, rigid bodies, weightless springs, etc.) and oscillating components with distributed parameters (strings, rods, elastic beams and shafts, membranes, plates, cavities with stratified liquid, etc.), whose frequency spectrum is denumerably infinite. The controlling actions, whether kinematic or dynamic in nature, are assumed to be concentrated with respect to the space variables. They may be movable, applied to the absolutely rigid parts of the system and/or fixed at the boundaries of the distributed elements (boundary control). This kind of control is of value in applications. On the assumption that techniques of mathematical physics /1, 2/ or the method of moments /3, 4/ yield an infinite-dimensional control problem for the Fourier coefficients of the solution relative to a set of basis functions for the boundary-value problem, an asymptotic approach is proposed for constructing approximate controls and the resulting scheme for the approximate solution of the problem is shown to be legitimate. A specific problem is examined as an illustration - rotation of an elastic rod in the plane by a torque applied at its end.

1. *Preliminary assumptions and statement of the problem.* As it is not our intention to study mechanical controlled systems in all their generality, attention will be confined to scalar control functions $w(t)$ /4-9/. It is assumed that the Fourier coefficients $s_i(t)$ of an unknown distribution $z(t, x)$, relative to a (given) basis $\{r_i(x)\}$ which is orthonormal with weight $\rho(x)$, satisfy a denumerably infinite set of equations of the following form:

$$s_i^{\cdot\cdot} + \omega_i^2 s_i = \alpha_i w, \quad i = 0, 1, 2, \ldots, n, \ldots; \quad t \in [0, T] \tag{1.1}$$
$$0 \leqslant \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_n < \cdots, \quad \alpha_i \neq 0$$
$$z(t, x) = \sum_{i=0}^{\infty} s_i(t) r_i(x), \quad x \in D \subset R^p \quad (r_i, r_j)_\rho = \delta_{ij}$$

The vector $(s, s^\cdot)$ is treated as an element of denumerable-dimensional Euclidean space. The natural frequencies of the oscillations ω_i, $i \geqslant 1$, are assumed to be simple, and the influence parameters α_i characterizing the efficiency of the control in the i-th mode do not vanish; $\dim D = p \geqslant 1$.

In the distributed oscillating systems encountered in practice, the frequencies ω_i, as functions of the discrete parameter i, may exhibit the following types of asymptotic behaviour as $i \to \infty$:

1) $\omega_i \sim \sqrt{i}$, $\omega_{i+1} - \omega_i \sim 1/\sqrt{i}$ (gravitational waves in a cavity containing a homogeneous liquid with free surface or a discretely stratified liquid /9/);

2) $\omega_i \sim i$, $\omega_{i+1} - \omega_i \sim 1$ (transverse waves in a string or elastic shaft; longitudinal waves in a distributed spring or elastic beam /4, 7/);

3) $\omega_i \sim \sqrt{i^3}$, $\omega_{i+1} - \omega_i \sim \sqrt{i}$ (gravitational-capillary or capillary waves in a cavity like that of case 1);

4) $\omega_i \sim i^2$, $\omega_{i+1} - \omega_i \sim i$ (transverse waves in an elastic rod or shaft /8/).

The initial values $\{s_i^0, v_i^0\}$ of the variables $\{s_i, v_i\}$, $i \geqslant 0$ $(v_i(t) \equiv s_i^\cdot(t))$ at $t = 0$ are obtained in the standard way as the Fourier coefficients of the initial distributions of the displacements $z(0, x)$ and velocities $z^\cdot(0, x)$, expanded in terms of the basis $\{r_i(x)\}$ generated by an appropriate selfadjoint boundary-value problem /1, 2/. These distributions

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The so-called finite control problem /4/ is as follows: construct a function $w(t)$ from some admissible class W $(w(t) \in W)$ that will steer system (1.1) from state $\{s_i^0, v_i^0\}$ to a designated state $\{s_i^T, v_i^T\}$, $i \geqslant 0$, at $t = T$, $T < \infty$. The admissible class W is determined by such considerations as physical realizability and the existence of a sufficiently smooth solution $z = z(t, x)$. It is commonly assumed that $w(t)$ is square integrable, i.e., $w(t) \in W \subseteq L_2[0, T]$, and that the series (1.1) for the solution $(z(t, x), z^{\cdot}(t, x))$ is convergent in the norm of the energy space (the energy of the system must be finite for all $t \in [0, T]$, $T < \infty$) /1, 2, 5/. Thus, we have to solve the following two-point problem with respect to t:

$$s_i(0) = s_i^0, \quad v_i(0) = v_i^0; \quad s_i(T) = s_i^T, \quad v_i(T) = v_i^T \tag{1.2}$$

Note that the terminal distribution functions of the displacements and the velocities defined by the coefficients $\{s_i^T, v_i^T\}$ must satisfy the boundary conditions. In a more general case, the boundary conditions may stipulate the values of finitely or denumerable many functionals at $t = T$ /7/; in particular, the boundary conditions may pertain to only some of the coefficients s_i^T, v_i^T ($i \in I$, where I is some index set). The following special cases are of obvious mechanical interest.

1. $\omega_0 = 0$ and $s_0(T) = s_0^T$, $v_0(T) = v_0^T$, $s_i(T) = v_i(T) = 0$ - the system is brought as a whole to a state of motion; $z(t, x) = s_0(t) r_0(x)$, $t \geqslant T$, where $s_0(t) = s_0^T + v_0^T(t - T)$, with relative oscillations suppressed. If it is also required that the amplitude of the partial oscillations be such that $A_{i*}^T = [(s_{i*}^T)^2 + (v_{i*}^T / \omega_{i*})^2]^{1/2} \neq 0$, the system will move as a whole and perform monochromatic oscillations at a frequency ω_{i*}.

2. If $s_i(T) = s_i^T$ but $v_i(T)$ $(i \geqslant 0)$ are arbitrary, this is the problem of "stiff" control; if it is required that $A_i^T = 0$ $(i \geqslant 1,\ \omega_0 = 0)$, but s_0^T, v_0^T are arbitrary, this is the problem of suppression of relative oscillations. If s_i^T, v_i^T, $i = 0, 1, 2, \ldots, n$, are prescribed, this is known as an n-mode approximation of the control problem; if $n = 0$, i.e., only s_0^T, v_0^T are prescribed, this is the "coarse" formulation of the problem, since no attention is paid to the relative oscillations. But if $\omega_0 > 0$, the control system is purely oscillatory.

The controllability problem for a denumerable oscillating system (1.1), (1.2) in a finite time interval, with a finite number of control functions (in particular, one control $w(t)$), involves considerable theoretical difficulties /4-9/. The controls obtained are more often than not generalized functions. Instead, one can adopt a constructive approach to the corresponding approximate control problem: for example, one might require the terminal Conditions (1.2) to be satisfied only to within a prescribed error $O(\varepsilon)$ $(0 < \varepsilon \ll 1)$.

In this paper we propose an approach to the approximate solution of the control problem over an asymptotically large time interval, determined by a small parameter ε in the following way:

$$t \in [0, T], \ T = \Theta\varepsilon^{-1}, \ \varepsilon \in (0, \varepsilon_0], \ \Theta = O(1) \tag{1.3}$$

The coefficients α_i $(i \geqslant 0)$ in (1.1) are of the order of unity relative to the parameter ε (i.e., $(\alpha_i \sim 1)$; as i increases they may either decrease or increase, remaining bounded or not. The frequencies ω_i $(i \geqslant 1)$ in (1.1) are also of order unity relative to ε; the typical situation was described previously. The quantities s_0^0, s_0^T and $(s_0^0 - s_0^T)$ may be asymptotically large, but the amplitudes of the partial oscillations and the velocity $v_0^{0,T}$, i.e., the energy of the system, are of the order of unity. In sum, we are making the following assumptions $(\omega_0 = 0)$:

$$\begin{gathered} s_0^{0,T} \sim 1/\varepsilon, \quad (s_0^0 - s_0^T) \sim 1/\varepsilon, \quad v_0^{0,T} \sim 1 \\ A_j^{0,T} = [(s_j^{0,T})^2 + (v_j^{0,T}/\omega_j)^2]^{1/2} \sim 1, \quad A_\Sigma^{0,T} = \Big[\sum_{j=1}^{\infty} (A_j^{0,T})^2\Big]^{1/2} \sim 1 \\ E_i^{0,T} = {}^1/_2(\omega_i A_i^{0,T})^2 \sim 1, \quad E_\Sigma^{0,T} = \sum_{i=0}^{\infty} E_i^{0,T} \sim 1 \quad \Big(E_0 = \frac{1}{2} v_0^2\Big) \end{gathered} \tag{1.4}$$

Here E_i $(i \geqslant 0)$ are the total "energies" of the partials and E_Σ is the total "energy" of the system.

Note that throughout the control process the values of $s_i(t)$, $v_i(t)$ $(i \geqslant 0)$ for $\forall t \in [0, T]$, where $T \sim \varepsilon^{-1}$, satisfy Condition (1.4), subject to an appropriate choice of the control function $w(t)$.

System (1.1) will be considered throughout the sequel in the case $\omega_0 = 0$, $\alpha_0 = 1$ and interpreted as a denumerable set of linear unidirectional oscillators (or two-dimensional pendulums on parallel axes) on a common base /6, 7/. A control force $w(t)$ of kinematic or dynamic nature is applied to the base. In the first case $s_0^{\cdot\cdot} = w$ is the acceleration of a certain point of the base, in the second it is the acceleration of the system's centre of mass.

The controllability of a denumerable system of pendulums has been studied in /4-9/ and elsewhere.

2. Approximate solution: motion of a denumerable system of oscillators (pendulums) with suppression of relative oscillations. To fix our ideas (without much loss of generality) we shall consider the problem of steering system (1.1) at time $t = T$ (1.3) to a prescribed state of motion as a whole without relative oscillations, i.e., the terminal Conditions (1.2) and required motion at $t \geqslant T$ are assumed to be

$$\begin{gathered} s_0(T) = s_0^T \ (=0), \ \ v_0(T) = v_0^T \ (=0), \ \ s_i(T) = v_i(T) = 0 \\ \forall t > T, \ w(t) \equiv 0: \ s_0(t) = s_0^T + v_0^T(t-T), \ v_0(t) = v_0^T \\ s_i(t) = v_i(t) \equiv 0 \quad (i = 1, 2, \ldots, n, \ldots; \ T = \Theta\varepsilon^{-1}) \end{gathered} \tag{2.1}$$

It may be assumed here that $s_0^T = v_0^T = 0$; this is achieved by transforming to variables $s_0^* = s_0 - s_0^T$, $v_0^* = v_0 - v_0^T$ (and doing the same in the initial conditons). The required motion is then the rest state: $z(t, x) = z^\cdot(t, x) \equiv 0, x \in D, t \geqslant T$.

The required control $w(t) \in W$ for Problem (1.1), (1.2), (2.1) is assumed to be a series

$$\begin{gathered} w = w(t, \Pi) \equiv a_0 t + b_0 + \sum_{j=1}^{\infty} (a_j \sin \omega_j t + b_j \cos \omega_j t) \\ \Pi = \{a_i, b_i\}, \ i = 0, 1, \ldots, n, \ldots; \ \Pi = \text{const} \end{gathered} \tag{2.2}$$

where $\Pi \in l_2$ is an unknown vector in denumerable-dimensional Euclidean space l_2, to be determined in accordance with conditions (2.1). Note that a series of type (2.2) can be obtained from the expression for the optimal control of system (1.1) with an integral quadratic performance index /4, 5, 7/:

$$J[w] = \frac{1}{2} \int_0^T w^2(t)\, dt \to \min_{|w| < \infty}, \quad w(t) \in W \subseteq L_2[0, T] \tag{2.3}$$

Substituting the series (2.2) for $w(t, \Pi)$ into system (1.1) and integrating, using the initial Conditions (1.2), we obtain the following representation for the solution as a Cauchy problem:

$$\begin{gathered} s_0 = s_0(t, \Pi) = s_0^0 + v_0^0 t + \frac{a_0}{6} t^3 + \frac{b_0}{2} t^2 + \sum_{j=1}^{\infty} \frac{a_j \varphi_j}{\omega_j^2} + S_0(t, \Pi) \\ v_0 = v_0(t, \Pi) = v_0^0 + {}^1/_2 a_0 t^2 + b_0 t + V_0(t, \Pi) \quad (\varphi_i = \omega_i t) \\ s_i = s_i(t, \Pi) = s_0^0 \cos \varphi_i + (v_i^0/\omega_i) \sin \varphi_i - \\ {}^1/_2 (\alpha_i a_i \varphi_i/\omega_i^2) \cos \varphi_i + {}^1/_2 (\alpha_i b_i \varphi_i/\omega_i^2) \sin \varphi_i + S_i(t, \Pi) \\ v_i = v_i(t, \Pi) = -s_i^0 \omega_i \sin \varphi_i + v_i^0 \cos \varphi_i + \\ {}^1/_2 (\alpha_i a_i \varphi_i/\omega_i) \sin \varphi_i + {}^1/_2 (\alpha_i b_i \varphi_i/\omega_i) \cos \varphi_i + V_i(t, \Pi) \end{gathered} \tag{2.4}$$

We have written out these expressions with the "principal" terms $O(1/\varepsilon)$ and $O(1)$ separated from the small perturbations $O(\varepsilon)$ for $\forall t \in [0, T]$, $T = \Theta\varepsilon^{-1}$; the latter are represented by the terms S_i, V_i $(i = 0, 1, \ldots)$. In regard to this separation it should be noted that $V_i(t, \Pi) = S_i^\cdot(t, \Pi)$, $i \geqslant 1$, but $V_0(t, \Pi) \neq S_0^\cdot(t, \Pi)$. This representation depends on the assumption that the series representing S_i, V_i are convergent:

$$\begin{gathered} S_0(t, \Pi) = \sum_{j=1}^{\infty} \frac{1}{\omega_j^2} [-a_j \sin \varphi_j + b_j(1 - \cos \varphi_j)] \\ V_0(t, \Pi) = \sum_{j=1}^{\infty} \frac{1}{\omega_j} [a_j(1 - \cos \varphi_j) + b_j \sin \varphi_j] \equiv S^\cdot(t, \Pi) + \sum_{j=1}^{\infty} \frac{a_j}{\omega_j} \\ S_i(t, \Pi) = \frac{\alpha_i a_0}{\omega_i^3} (\varphi_i - \sin \varphi_i) + \frac{\alpha_i b_0}{\omega_i^2} (1 - \cos \varphi_i) + \frac{\alpha_i a_i}{2\omega_i^2} \sin \varphi_i + \\ \frac{\alpha_i}{\omega_i} {\sum_{j=1}^{\infty}}' \left(a_j \frac{\omega_i \sin \varphi_j - \omega_j \sin \varphi_i}{\omega_i^2 - \omega_j^2} - b_j \omega_i \frac{\cos \varphi_i - \cos \varphi_j}{\omega_i^2 - \omega_j^2} \right) \\ V_i(t, \Pi) = S_i^\cdot(t, \Pi), \ i \geqslant 1, \ t \in [0, \Theta\varepsilon^{-1}] \end{gathered} \tag{2.5}$$

The symbol Σ' means that the series omits terms with $j = i$ (which are included in the principal terms).

The denumerable set of parameters $\Pi = \{a_i, b_i\}$, $i \geqslant 0$, is determined by the zero terminal Conditions (2.1) for representations (2.4), (2.5), which yield linear equations for Π:

$$s_i(T, \Pi) = 0, \quad v_i(T, \Pi) = 0, \quad i = 0, 1, \ldots, n, \ldots \tag{2.6}$$

We will now solve system (2.6) approximately using the assumptions (see Sect.1) made concerning the order of magnitude of T, α_i, ω_i, s_i^0, v_i^0 relative to ε. Consider the following system of equations, which is a "first approximation" of system (2.6) with respect to ε, obtained by omitting terms assumed to be $O(\varepsilon)$:

$$\frac{a_0}{6}T^3 + \frac{b_0}{2}T^2 = -s_0^0 - v_0^0 T - \sum_{j=1}^{\infty}\frac{a_j}{\omega_j}\Phi_j, \quad \frac{a_0}{2}T^2 + b_0 T = -v_0^0 \tag{2.7}$$

$$a_i\cos\Phi_i - b_i\sin\Phi_i = \frac{2}{T}\frac{\omega_i}{\alpha_i}\left(s_i^0\cos\Phi_i + \frac{v_i^0}{\omega_i}\sin\Phi_i\right), \quad \Phi_i = \omega_i T$$

$$a_i\sin\Phi_i + b_i\cos\Phi_i = \frac{2}{T}\frac{\omega_i}{\alpha_i}\left(s_i^0\sin\Phi_i - \frac{v_i^0}{\omega_i}\cos\Phi_i\right), \quad i \geqslant 1$$

This is a diagonal system of equations for $\{a_i, b_i\}$, $i \geqslant 1$; solving it, we obtain first approximations of the coefficients $\{a_i^{(1)}, b_i^{(1)}\}$ with respect to ε. Substituting them into the first equation and combining the result with the second, we obtain a closed system of equations for (a_0, b_0). In sum we obtain the required values of the parameters Π:

$$a_0^{(1)} = (6 / T^3)(2s_0^0 + v_0^0 T + 4\xi), \quad a_i^{(1)} = 2\omega_i s_i^0 / (\alpha_i T) \tag{2.8}$$

$$b_0^{(1)} = -(2 / T^2)(3s_0^0 + 2v_0^0 T + 6\xi), \quad b_i^{(1)} = -2v_i^0 / (\alpha_i T)$$

$$\left(\xi = \sum_{j=1}^{\infty}\frac{s_j^0}{\alpha_j}, \quad \frac{s_j^0}{\alpha_j} = O(j^{-\gamma}), \quad \gamma > 1\right)$$

In view of our assumptions (1.4), it follows from these expressions that the coefficients a_i, b_i $(i \geqslant 0)$ are of different orders of magnitude relative to ε. If the series for ξ in (2.8) is convergent, we have

$$a_0^{(1)} = O(\varepsilon^2); \quad b_0^{(1)}, a_i^{(1)}, b_i^{(1)} = O(\varepsilon), \quad i \geqslant 1 \tag{2.9}$$

Substituting the values of $\Pi^{(1)}$ into (2.2), we obtain an expression for the control w, in the first approximation with respect to ε:

$$w^{(1)} = w(t, \Pi^{(1)}), \; \| w^{(1)} \| = O(\varepsilon), \quad t \in [0, \Theta\varepsilon^{-1}] \tag{2.10}$$

Similarly, an approximate controlled tarjectory $\{s_i(t, \Pi), v_i(t, \Pi)\}$, $i \geqslant 0$, is obtained by substituting $\Pi^{(1)}$ from (2.8) into (2.4) and omitting terms $O(\varepsilon)$, i.e., equating $S_i(t, \Pi) = V_i(t, \Pi) \equiv 0$.

3. ***Error estimate and the validity of the approximate solution.*** Let us assume that the coefficients $a_i^{(1)}$, $b_i^{(1)}$ satisfy the following very "generous" sufficient conditions:

$$\omega_i s_i^0/\alpha_i = O(i^{-\beta}), \quad v_i^0/\alpha_i = O(i^{-\beta}), \quad \beta > 1 \tag{3.1}$$

Since as a rule $\omega_i = O(i^{\varkappa})$, $\varkappa > 0$ (see Sect.1), the first of Conditions (3.1) is more restrictive than the first condition in (2.8). Then the series (2.10) is uniformly convergent to a continuous function $w(t, \Pi^{(1)})$, $t \in [0, \Theta\varepsilon^{-1}]$, satisfying this same estimate. In practice, control functions of this kind can be approximated fairly well by digital or analogue devices. Theoretically speaking, however, the class of controls $w(t, \Pi) \in W$ (2.2) can be enlarged, so as to consider controls for which the functions $\{s_i(t, \Pi^{(1)}), v_i(t, \Pi^{(1)})\}$, $i \geqslant 0$ (2.4) (as well as $\{z(t, x), z^{\cdot}(t, x)\}$ /see (1.1)/, and so on) are also admissible.

We will now compute the error involved in replacing the control by its first approximation $w^{(1)}$ (2.10), in terms of the values of s_i, v_i at $t = T$. By (2.4)-(2.8),

$$s_i(T, \Pi^{(1)}) = S_i(T, \Pi^{(1)}), \quad v_i(T, \Pi^{(1)}) = V_i(T, \Pi^{(1)}) \tag{3.2}$$

$$S_0(T, \Pi^{(1)}) = -\frac{2}{T}\sum_{j=1}^{\infty}\left[\frac{s_j^0}{\alpha_j\omega_j}\sin\Phi_j + \frac{v_j^0}{\alpha_j\omega_j^2}(1 - \cos\Phi_j)\right]$$

$$V_0(T, \Pi^{(1)}) = \frac{2}{T}\sum_{j=1}^{\infty}\left[\frac{s_j^0}{\alpha_j}(1 - \cos\Phi_j) - \frac{v_j^0}{\alpha_j\omega_j}\sin\Phi_j\right] \quad (i = 0)$$

$$S_i(T, \Pi^{(1)}) = \frac{a_0^{(1)}\alpha_i}{\omega_i^3}(\Phi_i - \sin\Phi_i) + \frac{b_0^{(1)}\alpha_i}{\omega_i^2}(1 - \cos\Phi_i) +$$

$$\frac{2}{T}\frac{\alpha_i}{\omega_i}\sum_{j=1}^{\infty}{}'\left(s_j{}^0\frac{\omega_j}{\alpha_j}\frac{\omega_i\sin\Phi_j-\omega_j\sin\Phi_i}{\omega_i^2-\omega_j^2}+v_j{}^0\frac{\omega_i}{\alpha_j}\frac{\cos\Phi_i-\cos\Phi_j}{\omega_i^2-\omega_j^2}\right)$$

$$V_i(T,\Pi^{(1)})=\partial S_i(T,\Pi^{(1)})/\partial T=S_i^{\cdot}(T,\Pi^{(1)})$$

The coefficients $a_0{}^{(1)}$, $b_0{}^{(1)}$ were defined in (2.8); when S_i is differentiated with respect to T to get (3.2), they are treated as constants (only Φ_i, Φ_j are differentiated; by (2.7), $\partial\Phi_{i,j}/\partial T=\omega_{i,j}$). If the series occurring in the expressions for S_i, V_i $(i\geqslant 1)$ in (3.2) are convergent - this is certainly the case if $s_j{}^0$, $v_j{}^0$ decrease sufficiently rapidly as j increases - we obtain the following error estimates:

$$|s_i(T,\Pi^{(1)})|\leqslant\varepsilon c_i{}^s,\quad |v_i(T,\Pi^{(1)})|\leqslant\varepsilon c_i{}^v,\quad i\geqslant 0 \tag{3.3}$$

The coefficients $c_i^{s,v}$ depend on the initial values $s_i{}^0$, $v_i{}^0$, the natural frequencies ω_j $(j\geqslant 1)$ and the influence coefficients α_j $(j\geqslant 0)$, as well as the parameter $\Theta\sim 1$. Incidentally, the series in (3.2) are certainly convergent in various special cases in which the initial distributions of displacements $z(0,x)$ and velocities $z^{\cdot}(0,x)$ involve only a finite number of harmonics, i.e., $A_j{}^0=0$ for $j>N$ $(N<\infty)$. If the initial relative displacements and velocities vanish (i.e., the distributed system is rigidly displaced), it follows from (3.2) that

$$s_0(T,\Pi^{(1)})=v_0(T,\Pi^{(1)})=0 \tag{3.4}$$
$$s_i(T,\Pi^{(1)})=O(\varepsilon^2),\quad v_i(T,\Pi^{(1)})=O(\varepsilon^2),\quad i\geqslant 1$$

In order to solve the initial problem approximately, in such a way as to determine the state variable $z(t,x)$ at $t=T$ to within a preassigned degree of accuracy in some metric (e.g., uniform, weak or strong), we must ensure that the parameters $c_i^{s,v}$ decrease sufficiently rapidly as $i\to\infty$. In addition, a similar condition must be imposed on the Fourier coefficients $s_i(t,\Pi^{(1)})$, $v_i(t,\Pi^{(1)})$ for $\forall t\in[0,T]$, so as to guarantee convergence (in the same norm) of the series for $z^{(1)}(t,x)$ and its derivatives with respect to t, x:

$$z^{(1)}(t,x)=\sum_{i=0}^{\infty}s_i(t,\Pi^{(1)})r_i(x),\quad\frac{\partial z^{(1)}}{\partial t},\quad\frac{\partial z^{(1)}}{\partial x},\ldots \tag{3.5}$$

For practical purposes, it is normally enough to guarantee convergence in the Hilbert or energy norm /1, 2, 4, 5/. In the special case of initial distributions with a finite number of modes, the general value (2.5) and terminal value (3.2), (3.3) of the coefficients S_i, V_i may be estimated as follows:

$$|S_0|\leqslant\varepsilon c,\quad |S_i|\leqslant\varepsilon c\,|\alpha_i|/\omega_i^2 \tag{3.6}$$
$$|V_0|\leqslant\varepsilon c,\quad |V_i|\leqslant\varepsilon c\,|\alpha_i|/\omega_i,\quad t\in[0,\Theta\varepsilon^{-1}],\quad c=\mathrm{const}$$

If the quotients $|\alpha_i|/\omega_i$ decrease sufficiently rapidly as i increases, the "remainder" terms" S_i, V_i in the formulae for s_i, v_i become small and may ultimately be omitted. The functions $z_{(1)}(t,x)$, $z^{\cdot}_{(1)}(t,x)$, $z'_{(1)}(t,x)$ etc. that result are ε-close to the functions $z^{(1)}(t,x)$, $\partial z^{(1)}/\partial t$, $\partial z^{(1)}/\partial x$ etc. in the appropriate norms (see (3.5)). A control (2.10) corresponding to this case will contain a finite number of terms of the series (2.2) $(1\leqslant j\leqslant N)$. It may also be expected that the approximate solution will be ε-close to the exact solution (in the same norms).

Remark 1. The solution of the problem with non-vanishing terminal values $s_i{}^T$, $v_i{}^T$, i.e., $A_i{}^T\neq 0$, $i\geqslant 1$, is obtained from the above solution by substituting

$$s_i{}^0\to s_i{}^0-(s_i{}^T\cos\Phi_i+(v_i{}^T/\omega_i)\sin\Phi_i)\quad(s_0{}^0\to s_0{}^0-s_0{}^T) \tag{3.7}$$
$$v_i{}^0\to v_i{}^0-(-s_i{}^T\sin\Phi_i+(v_i{}^T/\omega_i)\cos\Phi_i)\,\omega_i\quad(v_0{}^0\to v_0{}^0-v_0{}^T)$$

Remark 2. The problem of suppressing relative oscillations or controlling them without allowing for $s_0(T)$, $v_0(T)$ may be solved by putting $a_0^{(1)}=b_0^{(1)}=0$ in $\Pi^{(1)}$; the other coefficients $a_i^{(1)}$, $b_i^{(1)}$ $(i\geqslant 1)$ are given by (2.8), (3.7).

Remark 3. If it is desired to halt the system as a whole, i.e., $v_0(T) = A_i{}^T = 0$ (without allowing for $s_0(T)$), one must put $a_0^{(1)} = 0$, $b_0^{(1)} = v_0{}^0/T$, and the coefficients $a_i^{(1)}$, $b_i^{(1)}$ are computed from (2.8). To bring the system to a given position $s_0{}^T$ with non-fixed velocity $v_0(T)$ and suppression of relative oscillations, the requisite conditions are (2.8) for $a_i^{(1)}$, $b_i^{(1)}$, $i \geqslant 1$, and in addition

$$a_0^{(1)} = (3/T^3)(s_0{}^0 + v_0{}^0 T + 2\xi), \quad b_0^{(1)} = -a_0^{(1)} T$$

Analogous arguments yield approximate solutions for the control problem under other terminal conditions that lead to a diagonal system for (a_i, v_i), $i \geqslant 1$.

Remark 4. The parameter $\Pi = \{a_i, b_i\}$, $i \geqslant 0$, can be approximated to higher order in the small parameter ε by using (2.4)-(2.6) with S_i, V_i evaluated at Π values obtained in the previous stages of the recurrent procedure $(k = 1, 2, \ldots)$:

$$\begin{gathered}
\Pi_i^{(k+1)} = \Pi_i^{(1)} - P^{-1}(\Phi_i) R_i(T, \Pi^{(k)}), \quad \Pi_i = (a_i, b_i)^{\mathrm{tr}} \\
P^{-1} = P^{\mathrm{tr}}, \quad \det P = 1, \quad R_i = (S_i, V_i/\omega_i)^{\mathrm{tr}}; \quad A_i^{(k+1)} - A_i^{(k)} = O(\varepsilon^{k+1}) \\
a_0^{(k+1)} = a_0^{(1)} + (6/T^3)[2S_0(T, \Pi^{(k)}) - V_0(T, \Pi^{(k)})\, T] \\
b_0^{(k+1)} = b_0^{(1)} - (2/T^2)[3S_0(T, \Pi^{(k)}) - V_0(T, \Pi^{(k)}) T] \\
a_0^{(k+1)} - a_0^{(k)} = O(\varepsilon^{k+3}), \quad b_0^{(k+1)} - b_0^{(k)} = O(\varepsilon^{k+2})
\end{gathered} \tag{3.8}$$

Here $P(\Phi_i)$ is a 2×2 matrix representing rotation through the angle Φ_i. As to the convergence of the scheme (3.8), estimates of the error in terms of ε, expressions for the control $w(t, \Pi^{(k)})$ (2.2), the solution $z^{(k)}(t, x)$ and its derivatives (3.5) - all these problems involve considerable difficulties and merit separate discussion.

Remark 5. Quite naturally, one finds that synthesis of the control $w_s = w_s(T - t, s, v)$ is linear in the variables $s = (s_0, s_1, \ldots, s_n)^{\mathrm{tr}}$, $v = (v_0, v_1, \ldots, v_n, \ldots)^{\mathrm{tr}}$. To a first approximation with respect to ε, synthesis of the control by means of (2.2), (2.10), after substituting $t \to 0$, $T \to T - t$, $s^0 \to s$, $v^0 \to v$, gives

$$\begin{gathered}
w_s^{(1)} = w_s^{(1)}(T - t, s, v) \equiv \\
-2[3s_0 + 2v_0(T-t)]/(T-t)^2 - \frac{2}{(T-t)^2} \sum_{j=1}^{\infty} \left[6\frac{s_j}{\alpha_j} + \frac{v_j}{\alpha_j}(T-t) \right], \\
t \in \left[0, \frac{\theta}{\varepsilon}\right], \quad s_0 \sim \frac{1}{\varepsilon}; \quad v_0, A_\Sigma \sim 1
\end{gathered} \tag{3.9}$$

Note that this control synthesis is singular at $t \to T$ - a characteristic situation for time-independent control problems.

Remark 6. The above procedure for constructing approximate solutions and the accompanying error estimates are valid in more general systems of type (1.1). For example, the system may include a finite number or denumerably many groups of variables with multiple frequencies ω_j of finite multiplicity $k_j \leqslant k^*$, $k^* < \infty$, where w is a vector-valued control of the appropriate (sufficiently high) dimensionality and the controllability conditions are satisfied for each subgroup.

Remark 7. In applications one is sometimes interested in other settings of the control problem; for example, it may be required that the control be optimally fast. The major condition imposed on such controls is that they permit suppression of relative oscillations. Approaches similar to that outlined above, based on asymptotic separation of variables, may be applicable in the case when $T/T_1 = O(\varepsilon^{-1})$.

4. Analysis of controllable motion in the first approximation. By (2.4), (2.8), (3.6), variation of the oscillating variables s_i, v_i gives, to within $O(\varepsilon)$ $(i \geqslant 1)$:

$$\begin{gathered}
s_i^{(1)}(t) = A_i^{(1)}(\tau)\cos(\varphi_i - \varphi_i{}^0), \quad \cos\varphi_i{}^0 = s_i{}^0/A_i{}^0 \\
v_i^{(1)}(t) = -\omega_i A_i^{(1)}(\tau)\sin(\varphi_i - \varphi_i{}^0), \quad \sin\varphi_i{}^0 = v_i{}^0/(\omega_i A_i{}^0) \\
A_i^{(1)}(\tau) = A_i{}^0(1-\tau), \quad A_\Sigma^{(1)}(\tau) = A_\Sigma{}^0(1-\tau), \quad t/T = \tau \in [0, 1]
\end{gathered} \tag{4.1}$$

The behaviour of s_0, v_0 - the variables determining the motion of the system as a whole - is described to within an absolute error $O(\varepsilon)$ by the expressions

$$s_0{}^{(1)}(t) = s_0{}^0(1 - 3\tau^2 + 2\tau^3) + v_0{}^0 T(1-\tau)^2 + 2\xi\tau(1 - 3\tau + 2\tau^2) \tag{4.2}$$

$$v_0^{(1)}(t) = -6\,(s_0^0/T)\,\tau\,(1-\tau) + v_0^0\,(1 - 4\tau + 3\tau^2) + 2\,(\xi/T)(1 - 6\tau + 6\tau^2)$$

We first consider the global behaviour of $s_0^{(1)}, v_0^{(1)}$ with relative error $O(\varepsilon)$. To this end we normalize $s_0^{(1)}$ relative to $s_0^0 = O(\varepsilon^{(-1)})$ and $v_0^{(1)}$ relative to $s_0^0/T = O(1)$; this gives, again to within an error $O(\varepsilon)$,

$$\begin{gathered} s_0^{(1)}/s_0^0 \equiv s_0^{(*)}(\tau) = 1 - 3\tau^2 + 2\tau^3 + \nu\tau\,(1-\tau)^2 \\ v_0^{(1)}T/s_0^0 \equiv v_0^{(*)}(\tau) = -6\tau\,(1-\tau) + \nu\,(1 - 4\tau + 3\tau^2) \\ \nu = v_0^0 T/s_0^0 = O(1), \quad s_0^{(*)}(1) = v_0^{(*)}(1) = 0 \end{gathered} \tag{4.3}$$

It follows from (4.3) that if $\nu < -3$ or $\nu > -^3/_2$ there exists a point of inflection $\tau_* \in (0, 1)$ of $s_0^{(*)}(\tau)$ as a function of τ $(\tau_* = {}^1/_3\,(2\nu + 3)\,(\nu + 2)^{-1})$. At this point $v_0^{(*)}(\tau)$ has an extremum: if $\nu < -3$ it is a maximum, if $\nu > -^3/_2$ - a minimum. For all its simplicity, the family (4.3) (with ν as parameter, $v_0^{(*)}(0) = \nu$) is quite interesting, since it characterizes the quality of the control of the distributed system moving as a whole. Thus, when $\nu < -3$ one obtains supercontrol with respect to $s_0^{(*)}$ and $v_0^{(*)}$; when $\nu > -^3/_2$ the acceleration, i.e., $w^{(1)}$, changes sign, and the function $v_0^{(*)}(\tau)$ is not monotone; when $\nu > 0$ the function $s_0^{(*)}(\tau)$ is again non-monotone and $v_0^{(*)}(\tau)$ changes sign. The corresponding characteristic curves are shown in Fig.1 ($s_0^{(*)}(\tau)$) and Fig.2 ($v_0^{(*)}(\tau)$) (with the ν value indicated for each curve).

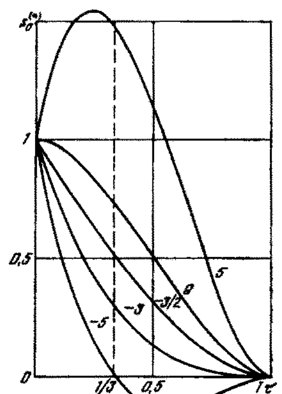

Fig.1

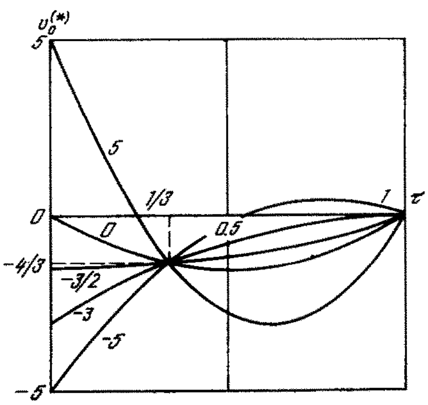

Fig.2

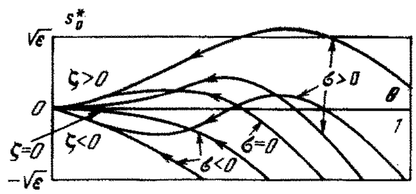

Fig.3

We now investigate the variables s_0, v_0 with absolute error $O(\varepsilon)$. It follows from (4.2) that if $(T - t) \sim 1$, i.e., $(1 - \tau) \sim \varepsilon$, then $s_0^{(1)}, v_0^{(1)} \sim \varepsilon$. If the time elapsing till the end of the process is asymptotically large, $(T - t) \sim 1/\sqrt{\varepsilon}$, i.e., $1 - \tau = \sqrt{\varepsilon}\theta$, where $\theta \in [0, \theta^*]$, $\theta^* \sim 1$, we obtain $(s_0^0 \sim \varepsilon^{-1})$

$$\begin{gathered} s_0^{(1)}/(\varepsilon s_0^0) \equiv s_0^*(\theta) = (3 + \nu)\,\theta^2 + \sqrt{\varepsilon}\,[(\nu - 2)\,\theta^3 + \zeta\theta] \\ v_0^{(1)}T/s_0^0 \equiv v_0^*(\theta) = -2\sqrt{\varepsilon}\,(3 + \nu)\,\theta \quad (\zeta = -2\xi/(\varepsilon s_0^0)) \end{gathered} \tag{4.4}$$

If $\nu \neq -3$ and ε is sufficiently small, s_0^* (and $s_0^{(1)}$) varies strongly (by $O(1)$), while v_0^* (and $v_0^{(1)}$) vary weakly (by $O(\sqrt{\varepsilon})$). For precision control to within $O(\varepsilon)$ one must allow for $O(\sqrt{\varepsilon})$ terms for $(T - t) \sim 1/\sqrt{\varepsilon}$. The trajectory in the (s_0^*, v_0^*) plane is approximately half of a strongly oblate parabola, corresponding to $v_0^*/(3 + \nu) < 0$.

If $\nu = -3 + \sqrt{\varepsilon}\sigma$, where $\sigma \sim 1$, then to within $O(\varepsilon)$

$$s_0^*(\theta) = \sqrt{\varepsilon}(\zeta\theta + \sigma\theta^2 - 5\theta^3), \quad v_0^*(\theta) \equiv 0, \quad \theta \geqslant 0 \tag{4.5}$$

The qualitative behaviour of $s_0^*(\theta)$ (4.5) for $\theta \sim 1$ $(\theta \to 0)$ for different values (signs) of ζ, σ is shown in Fig.3.

5. ***Control of two-dimensional rotations of an elastic rod.*** We consider the control of a straight inhomogeneous rod by a torque applied to its left end; the right end is free. The equations of motion may be derived by using d'Alembert's principle and the theorem on variation of the momentum; in the linear approximation /8/ we obtain

$$\rho(x) u^{\cdot\cdot} + [EI(x) u'']'' = -\rho(x) x\varphi^{\cdot\cdot}, \quad 0 < x < l \tag{5.1}$$

$$u(t, 0) = u'(t, 0) = u''(t, l) = u'''(t, l) = 0$$

$$\int_0^l \rho(x) x [x\varphi^{\cdot\cdot}(t) + u^{\cdot\cdot}(t, x)]\, dx = M, \quad M(t) \in \mathrm{M}$$

$$u = u(t, x), \quad \varphi = \varphi(t), \quad 0 \leqslant t \leqslant T < \infty, \ 0 \leqslant x \leqslant l$$

Here u is the relative displacement of a point x on the neutral line of the rod, in a system rotating together with the tangent to the rod at $x = 0$ (the point of application of the torque M. The dots denote differentiation with respect to t, and primes, with respect to x. The rotation of the tangent about some fixed axis is represented by the angle φ. The linear density $\rho(x)$ and bending rigidity $EI(x)$ are assumed to be sufficiently smooth, non-vanishing functions of x. The classes of controls $M(t) \in \mathbf{M}$ and solutions $u = u(t, x) \in U$ will be defined later.

As in Sect.1, we can consider the control problem for system (5.1). It is desired, by a suitable choice of an admissible control $M(t)$, to transfer the system from an arbitrary prescribed state at time $t = 0$ to a preassigned terminal state of rotation as a whole (with suppresssion of relative oscillations) at $t = T$:

$$u(0, x) = f^0(x), \quad u^{\cdot}(0, x) = g^0(x), \quad \varphi(0) = \varphi^0, \quad \varphi^{\cdot}(0) = \omega^0 \tag{5.2}$$

$$u(T, x) = f^T(x) \quad (\equiv 0), \quad u^{\cdot}(T, x) = g^T(x) \quad (\equiv 0)$$

$$\varphi(T) = \varphi^T \ (= 0), \quad \varphi^{\cdot}(T) = \omega^T \quad (= 0)$$

It is assumed that the corresponding selfadjoint boundary-value problem for eigenvalues and eigenfunctions has been solved:

$$[EI(x) X'']'' - \lambda^4 \rho(x) X = 0, \quad X = X(x), \quad 0 \leqslant x \leqslant l \tag{5.3}$$

$$X(0) = X'(0) = X''(l) = X'''(l) = 0$$

$$\lambda \in \{\lambda_n\}, \quad n = 1, 2, \ldots; \quad 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots, \ \lambda_n = O(n)$$

$$X(x) \in \{X_n(x)\}, \quad (X_n, X_m)_\rho \equiv \int_0^l X_n(x) X_m(x) \rho(x)\, dx = \delta_{nm}$$

The solution of Problem (5.1), (5.2) for $u(t, x)$ is obtained in terms of the complete orthonormal system of functions $\{X_n(x)\}$ (5.3) by the Fourier method /4, 7-9/:

$$u(t, x) = \sum_{n=1}^{\infty} X_n(x) \Theta_n(t), \quad \Theta_n = (u, X_n)_\rho \tag{5.4}$$

$$\Theta_n^{\cdot\cdot} + \omega_n^2 \Theta_n = -\mu_n \varphi^{\cdot\cdot}, \quad \omega_n = \lambda_n^2, \quad \mu_n = (x, X_n)_\rho, \quad n \geqslant 1$$

The equation of moments (4.1) may be reduced to the form $EI(0)\, u''(t, 0) = -M(t)$ using the equation of state and the boundary conditions; for given $M(t)$ it can be written as a Volterra equation of the first kind in $\gamma = \varphi^{\cdot\cdot}$:

$$\int_0^t G(t - \tau) \gamma(\tau)\, d\tau = N(t) + F(t), \quad t \in [0, T] \tag{5.5}$$

$$G(t) = \sum_{n=1}^{\infty} \frac{\mu_n}{\omega_n} X_n''(0) \sin \omega_n t, \quad N(t) = \frac{M(t)}{EI(0)}$$

$$F(t) = u_0''(t, 0)/(EI(0)); \quad u(t, x) = u_0(t, x) + u_\gamma(t, x)$$

$$u_0(t,x) = \sum_{n=1}^{\infty} X_n(x)\left(f_n^0 \cos\omega_n t + \frac{g_n^0}{\omega_n}\sin\omega_n t\right)$$

$$f_n^0 = (f^0, X_n)_\rho, \quad g_n^0 = (g^0, X_n)_\rho$$

$$u_\gamma(t,x) = -\sum_{n=1}^{\infty} \frac{\mu_n}{\omega_n} X_n(x) \int_0^t \sin\omega_n(t-\tau)\gamma(\tau)\,d\tau$$

The reader should note the need to coordinate the classes of functions $G(t)$, $N(t)$ and $F(t)$ occurring in Eq.(5.5) (see /10/).

Our approach leads to a control problem for the motion of a denumerable-dimensional system, described by differential Eq.(5.4) and integral Eq.(5.5). The technique of Sect.1-3 is not immediately applicable to this problem. Nevertheless, the solution can be computed approximately by the semi-inversion method, treating the function γ as the control and using the integral Eq.(5.5) to determine the required values of the torque $M(t)$ /8/:

$$\begin{gathered}\varphi^{\cdot} = \omega, \quad \omega^{\cdot} = \gamma, \quad \gamma(t) \in \Gamma \\ \Theta_n^{\cdot\cdot} + \omega_n^2\Theta_n = -\mu_n\gamma, \quad t \in [0, T], \quad n \geqslant 1 \\ \varphi(0) = \varphi^0, \quad \omega(0) = \omega^0, \quad \Theta_n(0) = f_n^0, \quad \Theta_n^{\cdot}(0) = g_n^0 \\ \varphi(T) = \varphi^T, \quad \omega(T) = \omega^T, \quad \Theta_n(T) = \Theta_n^{\cdot}(T) = 0\end{gathered} \tag{5.6}$$

Problem (5.6) is formally the same as that considered in Sects.1, 2, and if $T/T_1 \sim \varepsilon^{-1}$ $(0 < \varepsilon \ll 1,\ T_1 = 2\pi/\omega_1,\ \omega_1^2 \sim EI_*/(\rho_* l^4))$ the approximate solution procedure and analysis of Sects.2-4 are applicable. The constructions and formulae for the case of a homogeneous rod may be found in /8/. The controlling torque $M(t)$ computed from (5.5) is a more complicated function of t than (2.2) or (2.10). It involves the integral of the product of a function of that type and a kernel $G(t-\tau)$ which is an almost-periodic function with the same frequency basis $\{\omega_n\}$. As a result of the integration the function $M(t)$ involves products of linear functions of t and quasiperiodic functions.

We will now consider the direct computation of the control $M(t)$. To do this, we introduce a new variable - the "absolute" displacement of a point on the rod, $z = u + x\varphi$: this gives a controllable system for z /8/:

$$\begin{gathered}\rho(x) z^{\cdot\cdot} + [EI(x) z'']'' = 0, \quad z = z(t,x) = u(t,x) + x\varphi(t) \\ z(t,0) = z''(t,l) = z'''(t,l) = 0, \quad -EI(0) z''(t,0) = M(t) \\ \varphi(t) = z'(t,0), \quad u(t,x) = z(t,x) - xz'(t,0)\end{gathered} \tag{5.7}$$

The initial and terminal values of z may be computed on the basis of the "natural" Conditions (5.2) for u, φ and definition (5.7). These "generalized" conditions are

$$z(t,x)|_{0,T} = f^{0,T}(x) + \varphi^{0,T}x, \quad z^{\cdot}(t,x)|_{0,T} = g^{0,T}(x) + \omega^{0,T}x \tag{5.8}$$

Solution of Problem (5.7) with initial Conditions (5.8), for a known function $M(t)$, can be reduced by separation of variables and the Fourier method to the construction of the system of eigenvalues and eigenfunctions of a selfadjoint boundary-value problem similar to (5.3):

$$\begin{gathered}[EI(x) X'']'' - \lambda^4\rho(x) X = 0, \quad X = X(x), \quad 0 \leqslant x \leqslant l \\ X(0) = X''(0) = X''(l) = X'''(l) = 0 \\ \lambda \in \{\lambda_n\}, \quad 0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots; \ \lambda_n = O(n), \quad n \to \infty \\ X(x) \in \{X_n(x)\}, \quad (X_n, X_m)_\rho = \delta_{nm}, \quad n, m = 0, 1, 2, \ldots; \quad X_0 = x/\sqrt{J_0}\end{gathered} \tag{5.9}$$

Here the constant J_0 has the sense of the moment of inertia of an absolutely rigid rod. Suppose that the system of eigenvalues $\{\lambda_n\}$ and a complete system of orthonormal functions (basis) $\{X_n(x)\}$ have been constructed (for a uniform rod, see /8/). Then the required solution $z(t,x)$ is expressed as a Fourier series with coefficients $\Theta_n(t)$, $n \geqslant 0$:

$$z(t,x) = \frac{x}{\sqrt{J_0}}\Theta_0(t) + \sum_{n=1}^{\infty} X_n(x)\Theta_n(t) \quad \left(J_0 = \int_0^l x^2\rho(x)\,dx\right) \tag{5.10}$$

The unknown functions $\Theta_n(t)$, $n \geqslant 0$, are found by solving a system of differential equations of type (1.3) with initial and final Conditions (1.4), (1.5):

$$\Theta_0^{\cdot\cdot} = M/\sqrt{J_0}, \quad \Theta_n^{\cdot\cdot} + \omega_n^2\Theta_n = \mu_n M \quad (\mu_n = X_n'(0)) \tag{5.11}$$
$$\Theta_0|_{0,T} = \sqrt{J_0}\,\varphi^{0,T} - \sqrt{J_0}\sum_{n=1}^{\infty} X_n'(0) f_n^{0,T}, \quad \Theta_n|_{0,T} = f_n^{0,T} = (f^{0,T}, X_n)_\rho$$
$$\Theta_0^{\cdot}|_{0,T} = \sqrt{J_0}\,\omega^{0,T} - \sqrt{J_0}\sum_{n=1}^{\infty} X_n'(0) g_n^{0,T}, \quad \Theta_n^{\cdot}|_{0,T} = g_n^{0,T} = (g^{0,T}, X_n)_\rho$$

The solution of the two-point problem with respect to t and the choice of admissible control $M(t)$ proceed along the lines of Sects.2-4 above. In particular, for a uniform rod (ρ, $EI = \mathrm{const}$) we can introduce non-dimensional arguments: $x_* = x/l$, $t_* = \Omega t$, $\Omega^2 = EI/(\rho l^4)$; variables: $z_*(t_*, x_*) = z(\Omega^{-1}t_*, x_* l)/l$, $z_*^{\cdot}(t_*, x_*) = z^{\cdot}(\Omega^{-1}t_*, x_* l)/(l\Omega)$; and controlling torque: $M_* = M_*(t_*) = M(\Omega^{-1}t_*)\, l/(EI)$; finally, we obtain the solution (omitting the asterisk):

$$X_n(x) = \frac{\sin\lambda_n x}{\sin\lambda_n} + \frac{\operatorname{sh}\lambda_n x}{\operatorname{sh}\lambda_n} \quad (X_n, X_m) = \delta_{nm}$$
$$\operatorname{tg}\lambda = \operatorname{th}\lambda, \quad \lambda \in \{\lambda_n\}, \quad 0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots \tag{5.12}$$
$$\lambda_{n+1} = \pi n + \pi/4 + O(e^{-\pi n})$$
$$z(t, x) = 3x\int_0^t (t-\tau) M(\tau)\, d\tau +$$
$$\sum_{n=1}^{\infty} \frac{\operatorname{sh}\lambda_n \sin\lambda_n x + \sin\lambda_n \operatorname{sh}\lambda_n x}{\lambda_n(\operatorname{sh}\lambda_n - \sin\lambda_n)}\int_0^t \sin\omega_n(t-\tau) M(\tau)\, d\tau$$

Now, setting $M = M^{(1)}(t)$ (2.10), we obtain the required solutin to a first approximation with respect to ε, namely $z^{(1)}(t, x)$, in the form of (3.5), where $t \in [0, T]$, $T = 1/\varepsilon$, $0 < \varepsilon \ll 1$, $x \in [0, 1]$. If $M^{(1)}(t)$ is a piecewise-smooth function, the series in formula (5.12) for $z(t, x)$ is uniformly convergent, since it is majorized by a real number sequence whose terms decrease as n^{-3}; the series for $z'(t, x)$ is also absolutely and uniformly convergent, as a real sequence with terms $O(n^{-2})$. Convergence of higher-order derivatives with respect to x and t requires a separate investigation; this may be done in a non-uniform metric /1, 2, 5/.

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